

Symplectic Symmetry of the Neutrino Mass and the See-Saw Mechanism

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Abstract

We investigate the algebraic structure of the most general neutrino mass Hamiltonian and place the see-saw mechanism in an algebraic framework. We show that this Hamiltonian can be written in terms of the generators of an $Sp(4)$ algebra. The Pauli-Gürsey transformation is an $SU(2)$ rotation which is embedded in this $Sp(4)$ group. This $SU(2)$ also generates the see-saw mechanism.

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I. INTRODUCTION

Neutrinos play an essential role in electroweak physics and there is a fascinating interplay between neutrino properties and various astrophysical phenomena. Recent observations of especially the zenith-angle dependence of atmospheric neutrinos [1], and the solar neutrino flux [2] provide strong hints of non-zero neutrino masses and oscillations. Atmospheric and solar neutrino experiments along with direct experimental limits on the neutrino mass indicate that neutrino masses are much smaller than the charged-lepton masses. In the Standard Model neutrinos are left-handed. Hence the only Lorentz scalar pairing is the Majorana mass which is a weak ison triplet violating global lepton number conservation. Since the Standard Model has no ison triplet Higgs to couple to such a neutrino pair neutrinos are taken to be massless. Thus the neutrino mass is perhaps the most exquisite hint for physics beyond the Standard Model. In the current wisdom the Standard Model is not a fundamental, but an effective theory. Weak gauge bosons, W and Z , couple at low energies only with left-handed interactions. Neutrinos, being neutral leptons, interact only weakly at low energies. Hence the right-handed components of the neutrinos do not interact, that does not mean that they do not exist. A fundamental understanding of the origin and the magnitude of neutrino mass and mixings is lacking, however the see-saw mechanism [3] gives a natural scheme for the smallness of the neutrino masses and provides a convenient ansatz for model building. A very nice discussion of massive neutrino scenarios beyond the standard model is given in Reference [4].

Approaches to the neutrino mass based on symmetries are not fully exploited. Algebraic formulation of a problem could provide a check of the consistency of the formalism and may help to uncover symmetries. In particular it could be beneficial to rewrite the neutrino mass term in the Hamiltonian in terms of the elements of a Lie algebra which also generates the see-saw mechanism. The purpose of the present paper is to explore this possibility to place the neutrino mass matrix and the see-saw mechanism in the same algebraic framework.

One can view the mass term in the Hamiltonian as a generalized pairing problem. For a single neutrino family (four Dirac components) the most general pairing algebra is $SO(8)$ (see for example [5]). We show that for the neutrino mass Lorentz invariance constraints this algebra to be $Sp(4)$. We also show that the Pauli-Gürsey transformation is an $SU(2)_{\text{PG}}$ rotation which is embedded in the associated $Sp(4)$ Lie group. The see-saw Hamiltonian is generated by a particular Pauli-Gürsey transformation. The mass Hamiltonian sits in the $Sp(4)/SU(2)_{\text{PG}} \times U(1)$ coset where the $U(1)$ is a chirality transformation.

In the next section we introduce the $Sp(4)$ algebra for a single neutrino flavor and write the mass Hamiltonian in terms of its generators. In Section 3 we show that the Pauli-Gürsey transformation belongs to the associated group and obtain the see-saw mechanism as a particular Pauli-Gürsey transformation. We also discuss how the algebraic structure generalizes for the case of many flavors of neutrinos. Finally a brief discussion of our results conclude the paper.

II. THE ALGEBRAIC STRUCTURE OF THE NEUTRINO MASS MATRIX

We shall use the chiral representation of Dirac matrices, in which

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (2.1)$$

where the matrices $\vec{\sigma}$ are the usual 2×2 Pauli matrices. Since we want to introduce charges we will work with the Hamiltonians instead of Hamiltonian densities. Introducing the left- and right-handed fields

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi, \quad \psi_R = \frac{1}{2}(1 + \gamma_5)\psi \quad (2.2)$$

the Dirac-mass term in the Hamiltonian can be written as

$$H_m^D = m_D \int d^3x (\bar{\psi}_L \psi_R + h.c) \quad (2.3)$$

and the most general Majorana mass term as

$$H_m^M = H_m^L + H_m^R = \frac{1}{2}m_L \int d^3x (\bar{\psi}_L \psi_L^c + h.c) + \frac{1}{2}m_R \int d^3x (\bar{\psi}_R \psi_R^c + h.c). \quad (2.4)$$

In these equations m_D , m_L , and m_R are the Dirac, left-handed and right-handed Majorana masses and the charge conjugation of ψ field is defined by

$$\psi^C = C \bar{\psi}^T \quad (2.5)$$

where C is the charge conjugation matrix. In the chiral representation it is given by

$$C = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \quad (2.6)$$

Introducing the charges

$$D_+ = \int d^3x (\bar{\psi}_L \psi_R) \quad (2.7a)$$

$$D_- = \int d^3x (\bar{\psi}_R \psi_L) = D_+^\dagger \quad (2.7b)$$

the canonical equal-time anti-commutation relations between the fermion fields lead to

$$[D_+, D_-] = 2D_0 \quad (2.8)$$

where D_0 is defined as

$$D_0 = \frac{1}{2} \int d^3x (\psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R). \quad (2.9)$$

We see that D_+ , D_- and D_0 satisfy $SU(2)$ commutation relations:

$$[D_+, D_-] = 2D_0, \quad [D_0, D_+] = D_+, \quad [D_0, D_-] = -D_-. \quad (2.10)$$

Similarly introducing the left-handed

$$L_+ = \frac{1}{2} \int d^3x (\bar{\psi}_L \psi_L^c) \quad (2.11a)$$

$$L_- = \frac{1}{2} \int d^3x (\bar{\psi}_L^c \psi_L) = L_+^\dagger \quad (2.11b)$$

$$L_0 = \frac{1}{4} \int d^3x (\psi_L^\dagger \psi_L - \bar{\psi}_L \psi_L^\dagger) \quad (2.11c)$$

and the right-handed charges

$$R_+ = \frac{1}{2} \int d^3x (\bar{\psi}_R^c \psi_R) \quad (2.12a)$$

$$R_- = \frac{1}{2} \int d^3x (\bar{\psi}_R \psi_R^c) = R_+^\dagger \quad (2.12b)$$

$$R_0 = \frac{1}{4} \int d^3x (\psi_R^\dagger \psi_R - \bar{\psi}_R \psi_R^\dagger) \quad (2.12c)$$

one can see that they also satisfy $SU(2)$ algebra structure:

$$[L_+, L_-] = 2L_0, \quad [L_0, L_+] = L_+, \quad [L_0, L_-] = -L_- \quad (2.13)$$

$$[R_+, R_-] = 2R_0, \quad [R_0, R_+] = R_+, \quad [R_0, R_-] = -R_- \quad (2.14)$$

Also note that $D_0 \equiv L_0 + R_0$.

It is easy to show that the $SU(2)$ algebras generated by the left- and right-handed charges commute:

$$[L_{-,+,0}, R_{-,+,0}] = 0. \quad (2.15)$$

Calculating all the commutation relations between D 's, L 's and R 's to find a closed algebra structure provides a new set of generators:

$$[D_+, L_0] = -\frac{1}{2}D_+, \quad (2.16a)$$

$$[D_+, L_+] = 0, \quad (2.16b)$$

$$[D_+, L_-] = A_+, \quad (2.16c)$$

$$[D_+, R_0] = -\frac{1}{2}D_+, \quad (2.16d)$$

$$[D_+, R_+] = 0, \quad (2.16e)$$

$$[D_+, R_-] = A_-, \quad (2.16f)$$

$$[D_-, L_0] = \frac{1}{2}D_-, \quad (2.16g)$$

$$[D_-, L_+] = -A_-, \quad (2.16h)$$

$$[D_-, L_-] = 0, \quad (2.16i)$$

$$[D_-, R_0] = \frac{1}{2}D_-, \quad (2.16j)$$

$$[D_-, R_+] = -A_+, \quad (2.16k)$$

$$[D_-, R_-] = 0, \quad (2.16l)$$

where the additional generators are

$$A_+ = \int d^3x \left[-\psi_L^T C \gamma_0 \psi_R \right], \quad (2.17a)$$

$$A_- = \int d^3x \left[\psi_R^\dagger \gamma_0 C (\psi_L^\dagger)^T \right]. \quad (2.17b)$$

The commutation relations between them give another SU(2) algebra

$$[A_+, A_-] = 2(R_0 - L_0) \equiv 2A_0, \quad [A_+, A_0] = -A_+, \quad [A_-, A_0] = A_-. \quad (2.18)$$

Note that A_0 is proportional to the neutrino number operator. The commutation relations between A_+ and all the other generators

$$[A_+, D_+] = 2R_+, \quad [A_+, D_-] = -2L_-, \quad (2.19)$$

$$[A_+, L_0] = \frac{1}{2}A_+, \quad [A_+, L_+] = D_+, \quad [A_+, L_-] = 0, \quad (2.20)$$

$$[A_+, R_0] = -\frac{1}{2}A_+, \quad [A_+, R_+] = 0, \quad [A_+, R_-] = -D_- \quad (2.21)$$

along with the Hermitian conjugates of the commutators in Eqs. (2.19), (2.20), and (2.21) close the algebra: the ten independent generators; D_+ , D_- , L_+ , L_- , L_0 , R_+ , R_- , R_0 , A_+ , and A_- form a Lie algebra, the symplectic algebra $Sp(4)$.

In terms of these generators the most general mass term of the Hamiltonian can be written as

$$H_m = m_D(D_+ + D_-) + m_L(L_+ + L_-) + m_R(R_+ + R_-). \quad (2.22)$$

In writing Eq.(2.22) we ignored the phases of the masses. These can easily be incorporated as e.g. $m_L(e^{i\chi}L_+ + L_-e^{-i\chi})$.

One can write down a number of 4-dimensional matrix realizations of the $Sp(4)$ algebra. Here we present two of them. The first one can be expressed using the 2×2 Pauli matrices as :

$$D_0 = \frac{1}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad D_+ = \frac{1}{2} \begin{pmatrix} 0 & \sigma_+ \\ \sigma_+ & 0 \end{pmatrix}, \quad D_- = \frac{1}{2} \begin{pmatrix} 0 & \sigma_- \\ \sigma_- & 0 \end{pmatrix}, \quad (2.23)$$

$$\vec{L} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \vec{R} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.24)$$

$$A_+ = \frac{1}{2} \begin{pmatrix} 0 & \sigma_3 + 1 \\ \sigma_3 - 1 & 0 \end{pmatrix}, \quad A_- = \frac{1}{2} \begin{pmatrix} 0 & \sigma_3 - 1 \\ \sigma_3 + 1 & 0 \end{pmatrix}. \quad (2.25)$$

In these expressions 1 is the 2×2 unit matrix. The second matrix realization is given as

$$D_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_+ = \begin{pmatrix} 0 & \sigma_1 \\ 0 & 0 \end{pmatrix}, \quad D_- = \begin{pmatrix} 0 & 0 \\ \sigma_1 & 0 \end{pmatrix}; \quad (2.26)$$

$$L_0 = \frac{1}{4} \begin{pmatrix} \sigma_3 + 1 & 0 \\ 0 & -\sigma_3 - 1 \end{pmatrix}, \quad L_+ = \frac{1}{2} \begin{pmatrix} 0 & \sigma_3 + 1 \\ 0 & 0 \end{pmatrix}, \quad L_- = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ \sigma_3 + 1 & 0 \end{pmatrix}, \quad (2.27)$$

$$R_0 = \frac{1}{4} \begin{pmatrix} -\sigma_3 + 1 & 0 \\ 0 & \sigma_3 - 1 \end{pmatrix}, \quad R_+ = \frac{1}{2} \begin{pmatrix} 0 & -\sigma_3 + 1 \\ 0 & 0 \end{pmatrix}, \quad R_- = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -\sigma_3 + 1 & 0 \end{pmatrix}, \quad (2.28)$$

$$A_+ = \frac{1}{2} \begin{pmatrix} \sigma_- & 0 \\ 0 & -\sigma_+ \end{pmatrix}, \quad A_- = \frac{1}{2} \begin{pmatrix} \sigma_+ & 0 \\ 0 & -\sigma_- \end{pmatrix}. \quad (2.29)$$

Since there is only one four-dimensional representation of $Sp(4)$ these two matrix realizations can be transformed into one another by the unitary transformation

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.30)$$

Using the representation given in Eqs. (2.26), (2.27), (2.28), and (2.29) one can also write H_m of Eq. (2.22) in matrix form

$$H_m = \begin{pmatrix} 0 & 0 & m_L & m_D \\ 0 & 0 & m_D & m_R \\ m_L & m_D & 0 & 0 \\ m_D & m_R & 0 & 0 \end{pmatrix}. \quad (2.31)$$

Most discussions of the neutrino mass matrix start with Eq. (2.31) and its generalization to many flavors (see, for example Ref. [6]. A careful discussion which pays particular attention to the phases is given in Ref. [7]).

$Sp(4)$ is isomorphic to $Spin(5)$ which is the universal covering group of $SO(5)$. Representations of $Spin(5)$ can be considered as spinor representations of $SO(5)$. It may be

interesting to point out that these spinor representations can be realized using quaternions or, equivalently, using Pauli matrices. The four dimensional fundamental representation of $Sp(4)$ is explicitly given by Eqs. (2.26) through (2.29). To exhibit the $SU(2)$ -doublet structure of this representation we rewrite these equations as

$$D_{\pm} = \frac{1}{2} \sigma_1 \otimes \sigma_{\pm}, \quad (2.32)$$

$$\vec{L} = \frac{1}{4} (1 + \sigma_3) \otimes \vec{\sigma}, \quad (2.33)$$

$$\vec{R} = \frac{1}{4} (1 - \sigma_3) \otimes \vec{\sigma}, \quad (2.34)$$

and

$$A_{\pm} = \left[\sigma_{\mp} \otimes \frac{1}{4} (1 + \sigma_3) \right] \oplus \left[\sigma_{\pm} \otimes \frac{1}{4} (\sigma_3 - 1) \right]. \quad (2.35)$$

In these equations the notation $A \otimes \sigma_1$, etc. stands for the 4×4 block matrix where the elements of σ_1 are replaced by the 2×2 matrix A . Full details and the representation theory of spinor representations is covered in standard texts (see for example [8]).

III. THE PAULI-GÜRSEY TRANSFORMATION AND THE SEE-SAW MECHANISM

In 1957, W. Pauli introduced [9] a 3-parameter group which relates a Dirac spinor to its charge-conjugation. Later F. Gürsey showed [10] that this group can be extended to particles with mass and charge. The Pauli-Gürsey transformation is given by the 3-parameter transformation

$$\psi \rightarrow \psi' = a\psi + b\gamma_5\psi^c \quad (3.1)$$

where the parameters a and b satisfy the unitarity condition

$$|a|^2 + |b|^2 = 1. \quad (3.2)$$

In this section we elaborate on the relationship between the Pauli-Gürsey transformation and the neutrino mass matrix.

As the discussion in the previous section indicates the physical meaning of the $SU(2)$ algebras generated by the D 's, L 's, and R 's is clear. They generate the neutrino mass matrix. To illustrate the role of the $SU(2)$ algebra spanned by A_{\pm} and A_0 let us consider a general element of the associated $SU(2)$ group

$$\hat{U} = e^{-\tau^* A_-} e^{-\log(1+|\tau|^2) A_0} e^{\tau A_+} e^{i\varphi A_0}, \quad (3.3)$$

where τ and φ are arbitrary complex and real numbers respectively. Under this $SU(2)$ rotation the fermion field transforms into

$$\psi \rightarrow \psi' = \hat{U}\psi\hat{U}^\dagger = \frac{e^{i\varphi/2}}{\sqrt{1+|\tau|^2}}[\psi - \tau^*\gamma_5\psi^c]. \quad (3.4)$$

This is a Pauli-Gürsey transformation with the parameters

$$a = \frac{e^{i\varphi/2}}{\sqrt{1+|\tau|^2}}, \quad b = \frac{-\tau^*e^{i\varphi/2}}{\sqrt{1+|\tau|^2}}. \quad (3.5)$$

Note that if the mass term is considered a pairing interaction the Pauli-Gürsey transformation can be viewed as the associated Bogoliubov transformation. In light of Eqs. (3.4) and (3.5) from now on we designate the $SU(2)$ algebra spanned by A_\pm and A_0 as $SU(2)_{\text{PG}}$. It is also easy to show that the $U(1)$ algebra spanned by D_0 commutes with $SU(2)_{\text{PG}}$.

At this point the role of various components of the $Sp(4)$ algebra is identified. It is easy to show that

$$D_0 = -\frac{1}{2} \int d^3x (\psi^\dagger \gamma_5 \psi) \quad (3.6)$$

Hence the $U(1)_\chi$ algebra spanned by D_0 is a chirality transformation, the $SU(2)_{\text{PG}}$ generates the Pauli-Gürsey transformation and the most general neutrino mass Hamiltonian sits in the $Sp(4)/SU(2)_{\text{PG}} \times U(1)_\chi$ coset. It turns out that the $SU(2)_{\text{PG}}$ has another interesting function, namely that it generates the see-saw mechanism. To illustrate this we first rewrite Eq. (2.22) in the form

$$H_m = m_D(D_+ + D_-) + \left(\frac{m_L + m_R}{2}\right)[(L_+ + L_-) + (R_+ + R_-)] + \left(\frac{m_L - m_R}{2}\right)[(L_+ + L_-) - (R_+ + R_-)]. \quad (3.7)$$

One can easily show that under an $SU(2)_{\text{PG}}$ rotation one gets

$$L_+ \rightarrow L'_+ = \frac{e^{-i\varphi}}{1+|\tau|^2} [L_+ + \tau D_+ + \tau^2 R_+]; \quad (3.8)$$

$$R_+ \rightarrow R'_+ = \frac{e^{i\varphi}}{1+|\tau|^2} [R_+ - \tau^* D_+ + (\tau^*)^2 L_+]; \quad (3.9)$$

$$D_+ \rightarrow D'_+ = \frac{1}{1+|\tau|^2} [(1-|\tau|^2)D_+ + 2\tau R_+ - 2\tau^* L_+]. \quad (3.10)$$

From Eqs. (3.8) and (3.9) it follows that under an $SU(2)_{\text{PG}}$ rotation one has

$$[(L_+ + L_-) + (R_+ + R_-)] \rightarrow [(L'_+ + L'_-) + (R'_+ + R'_-)] = [(L_+ + L_-) + (R_+ + R_-)] \quad (3.11)$$

and

$$\begin{aligned} [(L_+ + L_-) - (R_+ + R_-)] &\rightarrow [(L'_+ + L'_-) - (R'_+ + R'_-)] \\ &= \frac{1}{1+\tau^2} [(1-\tau^2)((L_+ + L_-) - (R_+ + R_-)) + 2\tau(D_+ + D_-)]. \end{aligned} \quad (3.12)$$

In writing down Eqs. (3.11) and (3.12) we took $\varphi = 0$ and τ to be real assuming the masses in Eq. (3.7) have no phases. If phases are included one should restore φ and a complex τ .

Under the $SU(2)_{\text{PG}}$ rotation the mass Hamiltonian of Eq. (3.7) transforms into

$$\begin{aligned} H_m \rightarrow H'_m &= \hat{U} H_m \hat{U}^\dagger = m_D (D'_+ + D'_-) + \left(\frac{m_L + m_R}{2} \right) [(L'_+ + L'_-) + (R'_+ + R'_-)] \\ &\quad + \left(\frac{m_L - m_R}{2} \right) [(L'_+ + L'_-) - (R'_+ + R'_-)] \\ &= \left(\frac{m_L + m_R}{2} \right) [(L_+ + L_-) + (R_+ + R_-)] \\ &\quad + \frac{1}{1 + \tau^2} [m_D (1 - \tau^2) + (m_L - m_R) \tau] (D_+ + D_-) \\ &\quad + \frac{1}{1 + \tau^2} \left[-2m_D \tau + \left(\frac{m_L - m_R}{2} \right) (1 - \tau^2) \right] [(L_+ + L_-) - (R_+ + R_-)] \end{aligned} \quad (3.13)$$

One can eliminate various terms from Eq. (3.13) by a suitable choice of τ . For example, if one would like to eliminate the Dirac mass term $(D_+ + D_-)$ one can choose τ to make its coefficient zero. Introducing the angle δ

$$\tau = \tan \delta, \quad (3.14)$$

one finds that the choice

$$\tan 2\delta = -\frac{2m_D}{(m_L - m_R)}, \quad (3.15)$$

achieves this goal. The choice in Eq. (3.15) corresponds to:

$$\cos 2\delta = -\frac{(m_L - m_R)}{[4m_D^2 + (m_L - m_R)^2]^{1/2}}, \quad \sin 2\delta = \frac{2m_D}{[4m_D^2 + (m_L - m_R)^2]^{1/2}}, \quad (3.16)$$

Note that this solution exists even in the limiting case of $m_L = m_R$. Substituting Eq. (3.16) into Eq. (3.13) one obtains

$$\begin{aligned} H_m \rightarrow H'_m &= \hat{U} H_m \hat{U}^\dagger = \left(\frac{m_L + m_R}{2} \right) [(L_+ + L_-) + (R_+ + R_-)] \\ &\quad - \frac{1}{2} \left[4m_D^2 + (m_L - m_R)^2 \right]^{1/2} [(L_+ + L_-) - (R_+ + R_-)]. \end{aligned} \quad (3.17)$$

Making the choice $m_L = 0$ and $m_R \gg m_D$ one obtains the standard see-saw result

$$H'_m \approx m_R (R_+ + R_-) - \frac{m_D^2}{m_R} (L_+ + L_-). \quad (3.18)$$

So far the discussion was for a single neutrino flavor. For three neutrino flavors one can introduce three commuting copies of the $Sp(4)$ algebra and the similar arguments follow. It is possible to introduce a single $Sp(4)$ algebra for three flavors if the individual masses of the mass eigenstates are equal up to phases. This case seems to be too restrictive for model building as it is at variance with the recent observations.

IV. CONCLUSIONS

We showed that the neutrino mass problem can be formulated algebraically. The algebra is $Sp(4)$. A subgroup of this algebra generates the see-saw mechanism and the neutrino mass Hamiltonian sits in the leftover coset space. We should emphasize that even though in writing down the symplectic algebra we utilized a framework which is symmetric under the exchange of the left- and right-handed fields, the algebraic basis we introduced does not specify the values of m_L and m_R . In particular the value $m_L = 0$ is not excluded. Note that in the special case of $m_L = m_R$ and $m_D = 0$, Eq. (3.11) indicates that a general Pauli-Gürsey transformation given in Eq. (3.3) leaves the mass-term invariant. Indeed in this limit only the diagonal $SU(2)_{L+R}$ subgroup of $SU(2)_L \times SU(2)_R$ (spanned by the L 's and R 's respectively) is sufficient to describe the mass Hamiltonian. This $SU(2)_{L+R}$ commutes with $SU(2)_{PG}$.

Our analysis here concerns the neutrino mass, not the mixing matrix. Recent observations of the atmospheric neutrinos [1] very strongly suggest that muon neutrino maximally mixes with another neutrino. In this case one can show that one of the mass eigenstates decouples from the problem reducing the dimension of flavor space by one [11], which suggests the existence of an underlying symmetry. It would be interesting to see if the $Sp(4)$ or a higher symmetry could make a statement about the mixing matrix. Work along this direction is currently in progress.

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